CALCULUS
WORKSHEET ON OPTIMIZATION

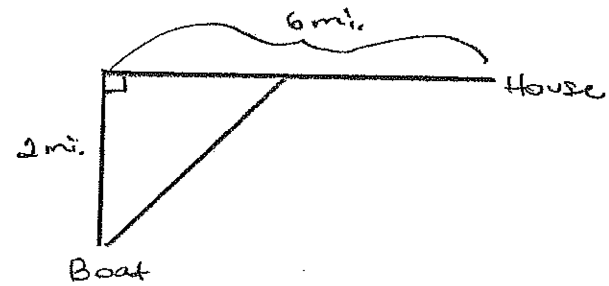
Work the following on **notebook paper**. Be sure to justify your answers.

1. A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

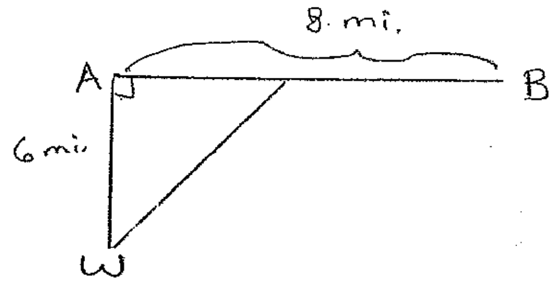
2. A tank with a rectangular base and rectangular sides is open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs \$10 per square meter for the base and \$5 per square meter for the sides, what is the cost of the least expensive tank, and what are its dimensions?

3. A cylindrical metal container, open at the top, is to have a capacity of 24π cubic inches. The cost of material used for the bottom of the container is \$0.15 per square inch, and the cost of the material used for the curved part is \$0.05 per square inch. Find the dimensions that will minimize the cost of the material, and find the minimum cost.

4. A person in a rowboat two miles from the nearest point on a straight shoreline wishes to reach a house six miles further down the shore. If the person can row at a rate of 3 miles per hour and walk at a rate of 5 miles per hour, find the least amount of time required to reach the house. How far from the house should the person land the rowboat?



5. An offshore well is located in the ocean at point W that is six miles from the closest shore point A on a straight shoreline. The oil is to be piped to a shore point B that is eight miles from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via a pipe along the shoreline. If the cost of laying pipe is \$100,000 per mile under water and \$75,000 per mile over land, how far from A should the point P be located to minimize the cost of lying the pipe? What will the cost be?



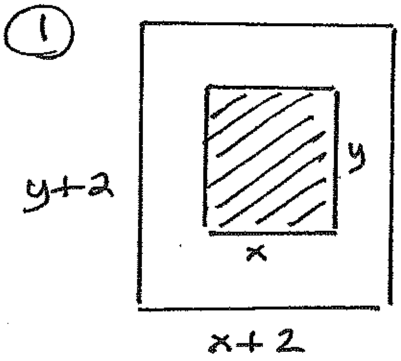
6. A piece of wire 40 cm long is to be cut into two pieces. One piece will be bent to form a circle, and the other will be bent to form a square. Find the radius of the circle and the length of the side of the square so that the total area is:
(a) a minimum
(b) a maximum

7. A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder The total volume of the solid is 12 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

8. An industrial tank of the shape described in problem 7 must have a volume of 3000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize the cost.

Worksheet

①



$$A = (x+2)(y+2)$$

$$A = (x+2)\left(\frac{30}{x}+2\right)$$

$$A = 30 + 2x + \frac{60}{x} + 4$$

$$A' = 2 - \frac{60}{x^2} = \frac{2x^2-60}{x^2} = 0 \text{ when } x = \sqrt{3}$$

$$xy = 30$$

$$y = \frac{30}{x}$$

$$\begin{array}{c|ccccc} A' & & - & & + & \\ \hline A & 0 & \searrow & \sqrt{30} & \nearrow & \end{array}$$

Min. when $x = \sqrt{30}$

so $y = \sqrt{30}$

Since A' changes from neg. to pos. to $x = \sqrt{30}$, A has its absolute minimum value at $x = \sqrt{30}$ so the dimensions of the page are $(2+\sqrt{30})$ in $\times$ $(2+\sqrt{30})$ in

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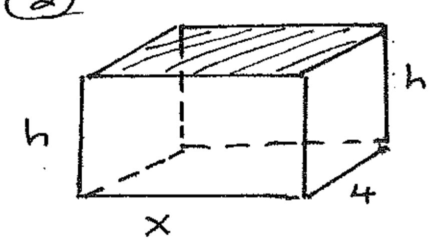
②



$$C = \$10(4x) + \$5(2)(hx) + \$5(2)(4h)$$

$$C = 40x + 10x\left(\frac{9}{x}\right) + 40\left(\frac{9}{x}\right)$$

$$C = 40x + 90 + \frac{360}{x}$$

$$C' = 40 - \frac{360}{x^2} = \frac{40x^2 - 360}{x^2}$$

$$C' = \frac{40(x^2-9)}{x^2} = 0 \quad \text{when } x = 3$$

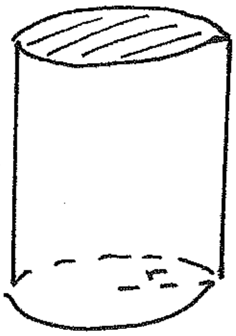
$$4xh = 36$$

$$h = \frac{9}{x}$$

$$\begin{array}{c|ccccc} C' & & - & & + & \\ \hline C & 0 & \searrow & 3 & \nearrow & \end{array}$$

Since C' changes from neg. to pos. at $x = 3$, C has its minimum value of \$330 when $x = 3$ m, $h = 3$ m so the dimensions are 3m × 3m × 4 m.

(3)



$$C = 15\pi r^2 + 5(2\pi r h)$$

$$C = 15\pi r^2 + 10\pi r\left(\frac{24}{r^2}\right)$$

$$C = 15\pi r^2 + \frac{240\pi}{r}$$

$$C' = 30\pi r - \frac{240\pi}{r^2} = \frac{30\pi r^3 - 240\pi}{r^2}$$

$$\pi r^2 h = 24\pi$$

$$h = \frac{24}{r^2}$$

$$C' = \frac{30\pi(r^3-8)}{r^2} = 0 \quad \text{when } r = 2$$

C': $-$ | $+$

C: 0 $\searrow$ 2 $\nearrow$

$\boxed{\text{Since } C' \text{ changes from neg. to pos. at } r=2 \text{ and } h=6,\ C \text{ has its minimum value of \$5.65 when } r = 2 \text{ in. and } h = 6 \text{ in.}}$

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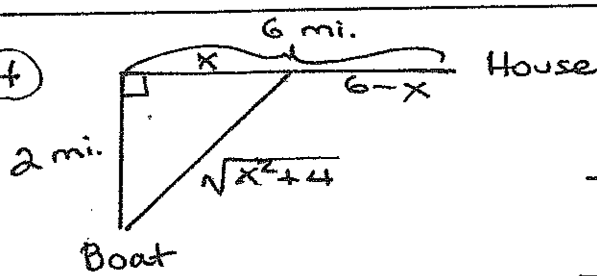
(4)



$$T = \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5}$$

$$T' = \frac{1}{3}\left(\frac{1}{2}(x^2+4)^{-1/2}(2x)\right) - \frac{1}{5}$$

$$0 \le x \le 6$$

$$T' = \frac{x}{3\sqrt{x^2+4}} - \frac{1}{5} = 0 \quad \text{when } x = 1.5$$

y1

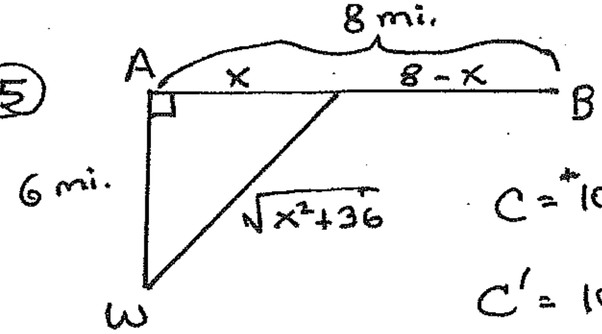
T': $-$ | $+$

T: 0 $\searrow$ 1.5 $\nearrow$ 6

$$\text{Time} = \frac{\text{Dist}}{\text{Rate}}$$

$\boxed{\text{Since } T' \text{ changes from neg. to pos. at } x = 1.5,\ T \text{ has its minimum value of } \underline{1.733 \text{ hr}} \text{ when } x = 1.5 \text{ mi. and the person lands the boat 4.5 miles from the house.}}$

(5)



$$C = \$100{,}000\sqrt{x^2+36} + \$75{,}000(8-x)$$

$$C' = 100{,}000\left(\tfrac{1}{2}(x^2+36)^{-1/2}(2x)\right) - 75{,}000$$

$$C' = \frac{100{,}000x}{\sqrt{x^2+36}} - 75{,}000 = 0$$

$y1 \rightarrow$ when $x = 6.803$

C': $-$ on $(0, 6.803)$, $+$ on $(6.803, 8)$

C: $0 \searrow 6.803 \nearrow 8$

Since C' changes from neg. to pos. at $x = 6.803$, C has its minimum value of \$996,862.70 when P is 6.803 mi. from A.

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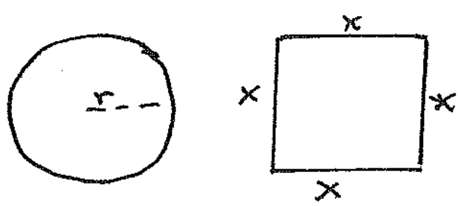
(6)



$\rightarrow 2\pi r + 4x = 40$

$4x = 40 - 2\pi r$

$x = 10 - \frac{1}{2}\pi r$

$0 \le r \le \frac{20}{\pi}$ $\qquad 0 \le x \le 10$

$$A = \pi r^2 + x^2$$

$$A = \pi r^2 + \left(10 - \tfrac{1}{2}\pi r\right)^2$$

$$A = \pi r^2 + 100 - 10\pi r + \tfrac{1}{4}\pi^2 r^2$$

$$A' = 2\pi r - 10\pi + \tfrac{1}{2}\pi^2 r = 0$$

$$\left(2\pi + \frac{\pi^2}{2}\right) r = 10\pi$$

$$r = \frac{10\pi}{2\pi + \frac{\pi^2}{2}} = \frac{20}{4+\pi} = 2.800$$

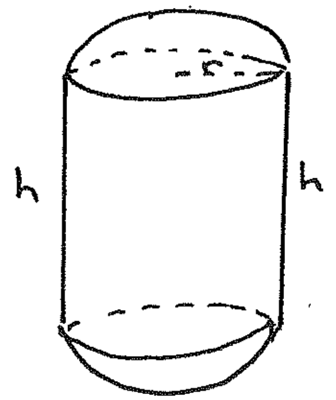
r	x	Area
0	10	100
$\frac{20}{4+\pi} = 2.800$	5.601	56.0099
$\frac{20}{\pi} = 6.366$	0	127.324

$$A' = 2x + \left(\frac{2}{\pi}\right)(-2)(20-2x)$$

(a) The minimum area is 56.010cm^2 when $r = \frac{20}{4+\pi} = 2.800$ cm and $x = \frac{40}{4+\pi} = 5.601$ cm.

(b) The maximum area is 127.324cm^2 when $r = \frac{20}{\pi}$ or 6.366 cm and $x = 0$ cm so use all of the wire for the circle.

(7)



$$A = 2\pi r h + 4\pi r^2$$

$$A = 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right) + 4\pi r^2$$

$$A = \frac{24}{r} - \frac{8}{3}\pi r^2 + 4\pi r^2$$

$$A = \frac{24}{r} + \frac{4}{3}\pi r^2$$

$$A' = \left(-\frac{24}{r^2} + \frac{8}{3}\pi r\right) = \boxed{\frac{8\pi r^3 - 72}{3r^2}} = 0$$

when $r = \sqrt[3]{\frac{9}{\pi}}$ or $1.420\ldots$

$$\pi r^2 h + \frac{4}{3}\pi r^3 = 12$$

$$\pi r^2 h = 12 - \frac{4}{3}\pi r^3$$

$$h = \frac{12 - \frac{4}{3}\pi r^3}{\pi r^2}$$

$$h = \frac{12}{\pi r^2} - \frac{4}{3}r$$

A'		$-$		$+$
A	0	$\searrow$	$1.420\ldots$	$\nearrow$

Since A' changes from neg. to pos. at $r = \sqrt[3]{\frac{9}{\pi}} = 1.420$ cm
A' is a minimum when $r = \sqrt[3]{\frac{9}{\pi}}$ or 1.420 cm.

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(8) $\pi r^2 h + \frac{4}{\pi} r^3 = 3000$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per sq. ft. of the surface area of the sides

$2k$ = cost per sq. ft. of the hemispherical ends

$$C = 2k(4\pi r^2) + k(2\pi r h)$$

$$C = 2k(4\pi r^2) + k(2\pi r)\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)$$

$$C = 8k\pi r^2 + \frac{6000k}{r} - \frac{8k\pi r^2}{3} = \frac{16k\pi r^2}{3} + \frac{6000k}{r}$$

$$C' = \frac{32k\pi r}{3} - \frac{6000k}{r^2} = \frac{32k\pi r^3 - 18{,}000k}{3r^2} = \frac{k(32\pi r^3 - 18{,}000)}{3r^2} =$$

when $r = \sqrt[3]{\frac{1125}{2\pi}} = 5.636$ ft

C'		$-$		$+$
C	0	$\searrow$	5.636	$\nearrow$

Since C' changes from neg. to pos. at $r = 5.636$ ft, the cost will be a minimum when $r = 5.636$ ft. and $h = 22.545$ ft.